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SYMMETRIC PERIODIC ORBITS OF THE MANY-BODY PROBLEM. RESONANCE AND PARADE OF PLANETS†

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The motion of a mechanical system consisting of $n+1$ material points attracting one another according to Newton's law is investigated. A reversible system of differential equations is derived for the motion of n points relative to the "main body". A small parameter is introduced. When this parameter is equated to zero, each of the n points is attracted by the "main body" only, and the generating system splits into n two-body problems. Two types of generating periodic orbits, symmetric about the fixed set M of an automorphism, are considered: (1) with both eccentricities and inclinations equal to zero; (2) with inclinations equal to zero.

It is shown that such orbits can be continued to non-zero values of the small parameter, as a result of which the system has periodic solutions of the first and second kinds. All these orbits are resonant: the mean motions of the bodies relate to one another as integers. In addition, at times that are multiples of the half-period the bodies are situated along a straight line, thus forming a "parade of planets".

The results also apply to a "Sun–planet–satellite" type system.

In the general theoretical part of the paper two methods are proposed for solving the problem of extending symmetric periodic motions to non-zero parameter values, and an upper bound is estimated for the domain of continuability.

1. SYMMETRIC PERIODIC SOLUTIONS OF THE REVERSIBLE SYSTEM

We consider a 2π-periodic system

$$\mathbf{u}^{\cdot} = \mathbf{U}_0(\mathbf{u},\mathbf{v},t)+\mu\mathbf{U}_1(\mu,\mathbf{u},\mathbf{v},t) \tag{1.1}$$

$$\mathbf{v}^{\cdot} = \mathbf{V}_0(\mathbf{u},\mathbf{v},t)+\mu\mathbf{V}_1(\mu,\mathbf{u},\mathbf{v},t), \qquad \mathbf{u}\in\mathbb{R}^l, \quad \mathbf{v}\in\mathbb{R}^n \quad (l\geq n)$$

and let $\mathbf{M}=\{\mathbf{u},\mathbf{v}\colon \mathbf{v}=\mathbf{0}\}$ be the fixed set of a linear automorphism of the system; μ is a small parameter. Suppose that when $\mu=0$ system (1.1) has a 2π-periodic solution

$$\mathbf{u}=\boldsymbol{\varphi}(t), \qquad \mathbf{v}=\boldsymbol{\psi}(t); \qquad \boldsymbol{\varphi}(-t)=\boldsymbol{\varphi}(t), \qquad \boldsymbol{\psi}(-t)=-\boldsymbol{\psi}(t)$$

which is symmetric about the set M. Our problem is to determine whether 2π-periodic symmetric solutions of (1.1) exist when $\mu\neq 0$. Such solutions are determined by the Heinbockel–Struble theorem [1]: if $\mathbf{u}^0$ and $\mathbf{v}^0$ are the values of $\mathbf{u}$ and $\mathbf{v}$ at $t=0$, then a sufficient condition for periodicity is the existence of solutions of the system of functional equations

$$\mathbf{v}^0=\mathbf{0}, \qquad \mathbf{v}(\mu,\mathbf{u}^0,\mathbf{v}^0,\pi)=\mathbf{0} \tag{1.2}$$

Suppose that when $\mu=0$ and $\mathbf{v}^0=0$ one has the condition $\det\|\partial v_s(0,\mathbf{u}^0,\mathbf{0},\pi)/\partial u_j\|\neq 0$. Then, by the implicit function theorem, a symmetric periodic solution has a unique continuation for small $\mu\neq 0$. Consequently, the possibility of continuation depends only on the generating system; sufficient conditions for continuation, including such conditions in cases that are not isolated in Poincaré's sense, have already been obtained [2].

Note that the limiting value of μ^* of the domain in which symmetric periodic motions can be continued for non-zero μ belongs to the domain. Otherwise the continuity of the $\|\mathbf{v}(\mu,\mathbf{u}^0,0,\pi)\|$ as a function of μ would be violated at $\mu=\mu^*$.

To solve specific mechanical problems, one proceeds as follows: Changing to perturbations $\mathbf{p}=\mathbf{u}-\boldsymbol{\varphi}(t)$, $\mathbf{q}=\mathbf{v}-\boldsymbol{\psi}(t)$ we obtain the equations

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$$
\begin{aligned}
\mathbf{p}^{\cdot} &= \mathbf{A}(t)\,\mathbf{p}+\mathbf{B}(t)\,\mathbf{q}+\mathbf{P}(\mathbf{p},\mathbf{q},t)+\mu\mathbf{U}(\mu,\mathbf{p}+\varphi(t),\mathbf{q}+\psi(t),t) \\
\mathbf{q}^{\cdot} &= \mathbf{C}(t)\,\mathbf{p}+\mathbf{D}(t)\,\mathbf{q}+\mathbf{Q}(\mathbf{p},\mathbf{q},t)+\mu\mathbf{V}(\mu,\mathbf{p}+\varphi(t),\mathbf{q}+\psi(t),t)
\end{aligned}
\tag{1.3}
$$

which are again reversible, with a set $\mathbf{M}_1 = \{\mathbf{p}, \mathbf{q}: \mathbf{q} = \mathbf{0}\}$ of fixed points of a linear automorphism. If one can construct [2] a fundamental system of solutions

$$
\mathbf{S}(t)=\begin{Vmatrix} \mathbf{p}^{+}(t) & \mathbf{p}^{-}(t) \\ \mathbf{q}^{-}(t) & \mathbf{q}^{+}(t) \end{Vmatrix}
$$

of the part of the equations that is linear in $\mathbf{p}$, $\mathbf{q}$ and independent of μ, then symmetric periodic solutions will be continuable provided that $\det \mathbf{q}^{-}(\pi) \neq 0$.

Another approach is convenient when the linear part can be reduced to a system with constant coefficients. In that case one can write [2]

$$
\begin{aligned}
&\xi^{\cdot} = \Xi(\mu,\xi,\eta,\zeta,t), \qquad \xi\in\mathbb{R}^{l-n} \\
&\eta^{\cdot}_{1,s} = H_{1,s}(\mu,\xi,\eta,\zeta,t), \qquad \zeta^{\cdot}_{1,s} = \eta_{1,s} + Z_{1,s}(\mu,\xi,\eta,\zeta,t) \\
&\eta^{\cdot}_{1,s} = \zeta_{j-1,s} + H_{j,s}(\mu,\xi,\eta,\zeta,t), \qquad \zeta^{\cdot}_{j,s} = \eta_{j,s} + Z_{j,s}(\mu,\xi,\eta,\zeta,t) \\
&\eta^{\cdot}_{1,\nu} = \kappa_\nu\zeta_{1,\nu} + H_{j,\nu}(\mu,\xi,\eta,\zeta,t), \qquad \zeta^{\cdot}_{1,\nu} = \kappa_\nu\eta_{1,\nu} + Z_{1,\nu}(\mu,\xi,\eta,\zeta,t) \\
&\eta^{\cdot}_{k+1,\nu} = \kappa_\nu\zeta_{k+1,\nu} + \zeta_{k,\nu} + H_{k+1,\nu}(\mu,\xi,\eta,\zeta,t) \\
&\zeta^{\cdot}_{k+1,\nu} = \kappa_\nu\eta_{k+1,\nu} + \eta_{k,\nu} + Z_{k+1,\nu}(\mu,\xi,\eta,\zeta,t)
\end{aligned}
\tag{1.4}
$$

and in the case of the automorphism $(t, \xi, \eta, \zeta) \to (-t, \xi, \eta, -\zeta)$ the solutions can be continued at all non-critical values $\kappa_\nu \neq Ni$ ($N \in \mathbb{Z}$). Clearly, this approach is especially useful if the linear part of system (1.3) does not depend explicitly on t.

In the analytic case, the domain of analyticity of the solution as a function of μ is determined by Poincaré's theorem [3]. Suppose that the power series on the right-hand sides of system (1.3) converge for $\|\mathbf{p}\|$, $\|\mathbf{q}\|$, $|\mu| \leqslant \alpha$, $|t| \leqslant \pi$. Then a solution of system (1.3) may be expanded in a power series in the initial values $\mathbf{p}^0$, $\mathbf{q}^0$ and the parameter μ, which converge in the domain

$$
\|\mathbf{p}^0\|, \quad \|\mathbf{q}^0\|, \quad |\mu| \leqslant \alpha^*, \quad |t|<\pi
\tag{1.5}
$$

The number α^* depends on α and on the Lipschitz constant L of system (1.3). It may be made as close to the number $\operatorname{Inf}\{\alpha/2, \alpha[\exp(\pi L) - 1]^{-1/2}\}$ as desired.

Within the domain (1.5), symmetric periodic solutions are constructed as series in powers of μ, with coefficients that depend on the initial data

$$
\begin{aligned}
\xi &= \xi_0(t)+\mu\xi_1(t)+\mu^2\xi_2(t)+\ldots \\
\eta &= \eta_0(t)+\mu\eta_1(t)+\mu^2\eta_2(t)+\ldots \\
\zeta &= \zeta_0(t)+\mu\zeta_1(t)+\mu^2\zeta_2(t)+\ldots
\end{aligned}
\tag{1.6}
$$

The terms $\xi_0(t)$, $\eta_0(t)$, $\zeta_0(t)$ characterize the perturbations of the generating system which, of course, may also have other symmetric periodic motions other than $\mathbf{p} = \mathbf{0}$, $\mathbf{q} = \mathbf{0}$.

As we are interested in the possibility of continuity this specific periodic solution ($\mathbf{p} = \mathbf{0}$, $\mathbf{q} = \mathbf{0}$), let us put $\xi_0(t) \equiv \mathbf{0}$, $\eta_0(t) \equiv \mathbf{0}$, $\zeta_0(t) \equiv \mathbf{0}$ in the series (1.6). Then the equations for the kth approximation will be

$$
\xi^{\cdot}_k = \mathbf{f}_k(t), \quad \eta^{\cdot}_k = \mathbf{G}\zeta_k + \mathbf{g}_k(t), \quad \zeta^{\cdot}_k = \mathbf{R}\eta_k + \mathbf{r}_k(t) \quad (k=1,2,\ldots)
\tag{1.7}
$$

where $\mathbf{f}_k(t)$, $\mathbf{g}_k(t)$, $\mathbf{r}_k(t)$ are 2π-periodic functions of t, which are known at the kth step, and the structure of the constant matrices $\mathbf{G}$ and $\mathbf{R}$ is clear from (1.4). Since the function $\varphi(t)$ is even and the function $\psi(t)$ odd for $k = 1$, it follows that $\mathbf{f}_1(t)$ and $\mathbf{g}_1(t)$ are odd functions of t, and $\mathbf{r}_1(t)$ is an even function of t. If the mean value of $\mathbf{r}_1(t)$ over a period is not zero, we add this number to the variable η_1 as an additive constant. Then the periodic solution defined by (1.7) consists of even functions $\xi_1(t)$, $\eta_1(t)$ and the odd function $\zeta_1(t)$. Under these conditions the solution always contains an arbitrary constant $\xi_1^0 = \xi_1(0)$; the initial value for ζ_1 is uniquely defined, and $\xi_1(0) = \mathbf{0}$.

The evenness (oddness) of the functions $\mathbf{f}_k(t)$, $\mathbf{g}_k(t)$, $(\mathbf{r}_k(t))$ for $k = 2, 3, \ldots$ follows immediately if one takes into account that the role of the functions $\varphi(t)$, $\psi(t)$ is now played by these functions plus the sum of the first $k-1$ terms of series (1.6).

Thus, the symmetric periodic motion (1.6) is represented by even functions $\xi(\mu, t)$, $\eta(\mu, t)$ and an odd function $\zeta(\mu, t)$. The solution with $\mathbf{p} = \mathbf{0}$, and $\mathbf{q} = \mathbf{0}$ determines a unique family of $1-n$ arbitrary constants ξ_1^0 and a parameter μ; the initial values ξ^0 and η^0 are represented by series in μ that converge for $|\mu| \leqslant \alpha^*$.

By Poincaré's theorem, the number α^* depends on the Lipschitz constant L, which can be defined as the upper bound of the moduli of the partial derivatives. This number depends on $|\kappa_\nu|$, and if the κ_ν are purely imaginary, all the moduli $|\kappa_\nu|$ may be made less than unity by applying the transformation $\eta_{j,\nu} + i\zeta_{j,\nu} \to (\eta_{j,\nu} + i\,\zeta_{j,\nu})\exp(ik_\nu t)$, $k_\nu \in \mathbb{Z}$. In addition, terms linear in μ and independent of ξ, η, ζ are cancelled out by an additional linear transformation, whose coefficients are 2π-periodic functions of t. This is equivalent to the definition of the functions $\xi_1(t)$, $\eta_1(t)$, $\zeta_1(t)$ in (1.6)

Thus, one can always assume that pure imaginary κ_ν in (1.4) are at most 1 in absolute value, while the not explicitly written out terms are non-linear in all of the variables μ, ξ, η, ζ.

2. EQUATIONS OF MOTION

Let us consider the fundamental problem of celestial mechanics—the motion of a mechanical system consisting of $n+1$ material points $\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_n$ with masses $m_0, m_1, \ldots, m_n$, respectively, attracting one another in accordance with Newton's law. The position of each point $\mathbf{M}_j$ ($j = 1, \ldots, n$) relative to $\mathbf{M}_0$ is determined by a number triple (ξ_j, η_j, ζ_j). Then the equations of motion of the system are

$$\begin{gathered}
\ddot{\xi}_s + \frac{k_s\xi_s}{r_{0s}^3} + \mu\sum_{\substack{j=1\\ j\neq s}}^{n} k_j^*\left[\frac{\xi_s-\xi_j}{r_{sj}^3} + \frac{\xi_j}{r_{0j}^3}\right] = 0 \\
\ddot{\eta}_s + \frac{k_s\eta_s}{r_{0s}^3} + \mu\sum_{\substack{j=1\\ j\neq s}}^{n} k_j^*\left[\frac{\eta_s-\eta_j}{r_{sj}^3} + \frac{\eta_j}{r_{0j}^3}\right] = 0 \qquad (2.1)\\
\ddot{\zeta}_s + \frac{k_s\zeta_s}{r_{0s}^3} + \mu\sum_{\substack{j=1\\ j\neq s}}^{n} k_j^*\left[\frac{\zeta_s-\zeta_j}{r_{sj}^3} + \frac{\zeta_j}{r_{0j}^3}\right] = 0
\end{gathered}$$

$r_{0s}^2 = \xi_s^2 + \eta_s^2 + \zeta_s^2$, $r_{sj}^2 = (\xi_s - \xi_j)^2 + (\eta_s - \eta_j)^2 + (\zeta_s - \zeta_j)^2$ $(s, j = 1, \ldots, n;\ s \neq j)$, where $k_s = k(1+\mu_s)$, $\mu = \max_j \mu_j$, $k_j^* = k\mu_j/\mu$, $\mu_j = m_j/m_0$, $k = fm_0$, and f is the gravitational constant.

The first term in the expression for the force in Eqs (2.1) characterizes the interaction of the bodies $\mathbf{M}_0$ and $\mathbf{M}_s$, the second, the influence of the bodies $\mathbf{M}_j$ ($j \neq s$) on the motion of $\mathbf{M}_s$. If the masses μ_s are small, then μ is small and this influence on the main problem—the motion of the two bodies $\mathbf{M}_0$ and $\mathbf{M}_s$—is small.

Another advantage of system (2.1) is its invariance with respect to certain linear transformations together with simultaneous time reversal, from t to $-t$. One such transformation is a change of sign in one of the groups of variables. Not wishing at this point to give a complete description of all linear automorphisms of system (2.1), we point out that the reversibility property enables one to prove the existence of symmetric periodic motions and to construct them.

3. PERIODIC SOLUTIONS OF THE FIRST KIND

If one transforms system (2.1) to cylindrical coordinates $(\boldsymbol{\rho}, \boldsymbol{\theta}, \boldsymbol{\zeta})$: $\xi_s = \rho_s\cos\theta_s$, $\eta_s = \rho_s\sin\theta_s$ ($s = 1, \ldots, n$), the equations of motion become

$$\ddot{\rho}_s - \rho_s\dot{\theta}_s^2 + \frac{k_s\rho_s}{(\rho_s^2+\zeta_s^2)^{3/2}} + \mu\sum_{\substack{j=1\\ j\neq s}}^{n} k_j^*\left[\frac{\rho_s}{r_{sj}^3} + \left(\frac{1}{r_{0j}^3} - \frac{1}{r_{sj}^3}\right)\rho_j\cos(\theta_s - \theta_j)\right] = 0$$

$$\frac{d}{dt}(\rho_s^2\theta_s^\cdot)+\mu\rho_s\sum_{\substack{j=1\\ j\neq s}}^{n}k_j^*\left[\frac{1}{r_{sj}^3}-\frac{1}{r_{0j}^3}\right]\rho_j\sin(\theta_s-\theta_j)=0 \tag{3.1}$$

$$\zeta_s^{\cdot\cdot}+\frac{k_s\zeta_s}{(\rho_s^2+\zeta_s^2)^{3/2}}+\mu\sum_{\substack{j=1\\ j\neq s}}^{n}k_j^*\left[\frac{\zeta_s-\zeta_j}{r_{sj}^3}+\frac{\zeta_j}{r_{0j}^3}\right]=0 \quad (s=1,\dots,n)$$

$$r_{sj}^2=\rho_s^2+\rho_j^2-2\rho_s\rho_j\cos(\theta_s-\theta_j)+(\zeta_s-\zeta_j)^2,\qquad r_{0j}^2=\rho_j^2+\zeta_j^2$$

If $\mu = 0$, we obtain the generating system which has a particular periodic solution

$$\rho_s=a_s(\text{const}),\quad \theta_s^\cdot=\omega_s(\text{const}),\quad \zeta_s=0,\quad \omega_s^2a_s^3=k_s\quad (s=1,\dots,n) \tag{3.2}$$

In this solution each of the bodies $\mathbf{M}_s$ moves in a circle of radius a_s at a constant angular velocity ω_s, all these circles lying in one fixed plane and having a common centre $\mathbf{M}_0$. As in the three-body problem [4], we will call periodic solutions arising from (3.2) when $\mu \neq 0$ solutions of the first kind.

Putting $\rho_s = a_s(1+p_s)$, $\theta_s = \omega_s t + \psi_s$ $(s = 1,\dots,n)$ in the neighbourhood of the generating solution, we obtain a system of equations for the variables ρ_s, ψ_s, ζ_s which is conditionally periodic in t, with frequencies $\{\omega_2-\omega_1,\dots,\omega_n-\omega_1\}$. This system will be periodic in time if $\omega_s-\omega_1 = l_s\omega$ $(\omega > 0)$, $l_s \in \mathbb{Z}$ $(s = 1,\dots,n)$. In that case, obviously, ω_s is not a multiple of ω if ω_1 is not a multiple of ω. If one now changes to a new dimensionless time variable $\tau = \omega t$ and a dimensionless variable $\zeta(\zeta_s \to a_s\zeta_s)$, the equations will be

$$p_s^{\cdot\cdot}-(1+p_s)(\omega_s/\omega+\psi_s^{\cdot})^2+\frac{\omega_s^2}{\omega^2}\frac{(1+p_s)}{[(1+p_s)^2+\zeta_s^2]^{3/2}}+$$

$$+\mu\sum_{\substack{j=1\\ j\neq s}}^{n}\frac{k_j^*}{\omega^2}\left\{\frac{1+p_s}{r_{sj}^3}+\left(\frac{1}{r_{0s}^3}-\frac{1}{r_{sj}^3}\right)\frac{a_j}{a_s}(1+p_j)\cos[(l_s-l_j)\tau+\psi_s-\psi_j]\right\}=0$$

$$\psi_s^{\cdot\cdot}+2\frac{\omega_s/\omega+\psi_s^{\cdot}}{1+p_s}p_s^{\cdot}+\frac{\mu}{(1+p_s)}\sum_{\substack{j=1\\ j\neq s}}^{n}\frac{k_j^*a_j}{\omega^2a_s}\left[\frac{1}{r_{sj}^3}-\frac{1}{r_{0j}^3}\right](1+p_j)\sin[(l_s-l_j)\tau+\psi_s-\psi_j]=0$$

$$\zeta_s^{\cdot\cdot}+\frac{\omega_s^2}{\omega^2}\frac{\zeta_s}{[(1+p_s)^2+\zeta_s^2]^{3/2}}+\mu\sum_{\substack{j=1\\ j\neq s}}^{n}\frac{k_j^*}{\omega^2}\left[\frac{\zeta_s}{r_{sj}^3}+\left(\frac{1}{r_{0j}^3}-\frac{1}{r_{sj}^3}\right)\frac{a_j}{a_s}\zeta_j\right]=0 \tag{3.3}$$

$$r_{sj}^2=a_s^2(1+p_s)^2+a_j^2(1+p_j)^2-2a_sa_j(1+p_s)(1+p_j)\cos[(l_s-l_j)\tau+\psi_s-\psi_j]+$$
$$+(a_s\zeta_s-a_j\zeta_j)^2,\qquad r_{0j}^2=a_j^2[(1+p_j)^2+\zeta_j^2]$$

where the prime denotes differentiation with respect to τ.

The explicit form of these equations enables us immediately to estimate the domain of analyticity, which is obviously determined by the possibility of expanding the reciprocals of the distances in series. In any case, the radius of convergence α may be any number not exceeding, say, $1/2$. Then the Lipschitz constant L depends on α and is determined from the partial derivatives. Once α has been chosen, L is a function of the parameters a_s, ω_1, ω, μ.

For this problem to be formally identical in all its detail with the problem of Section 1, let us assume that the change of variable $(\rho_s, \theta_s) \to (\rho_s, \theta_s^*)$, $\theta_s^* = \theta_s - \omega_1 T$, has been made in (3.1), thus transforming to a system of coordinates rotating uniformly at angular velocity ω_1. In these new variables one has $\theta_s^* = l_s\omega$, in the generating solution, while the fixed set of the automorphism is precisely the axis ξ_s^* $(\xi_s^* = \rho_s\cos\theta_s^*$, $\eta_s = \rho_s\sin\theta_s^*)$. Clearly, system (3.3) retains its previous form under this transformation.

System (3.3) is a 2π-periodic in τ. Therefore, if the linear approximation satisfies the conditions for the symmetric periodic solution (3.2) to be continuable to non-zero μ, such motions will exist at actual values of $\mu \leqslant \mu^*$. Bearing in mind that the application of these results to an actual Sun–planet system needs a special treatment, including the construction of periodic motions, we shall confine ourselves here to the qualitative aspect of the problem.

The part of system (3.3) which is linear in **p**, **ψ**, **ζ** and independent of μ is

$$p_s'' - 3\omega_s^{*2} p_s - 2\omega_s \psi_s' + \ldots = 0 \quad (\omega_s^* = \omega_1 / \omega + l_s) \tag{3.4}$$

$$\psi_s'' + 2\omega_s^* p_s' + \ldots = 0, \quad \zeta_s'' + \omega_s^{*2} \zeta_s + \ldots = 0 \quad (s = 1, \ldots, n)$$

An elementary transformation enables us to write the first two equations as

$$p_s'' + \omega_s^{*2} p_s - 2c_s + \ldots = 0, \quad \psi_s' = c_s - 2\omega_s^* p_s, \quad c_s' + \ldots = 0$$

Now, taking the automorphism $(\tau, \mathbf{p}, \mathbf{p}', \boldsymbol{\psi}, \boldsymbol{\psi}', \boldsymbol{\zeta}, \boldsymbol{\zeta}') \to (-\tau, \mathbf{p}, -\mathbf{p}', -\boldsymbol{\psi}, \boldsymbol{\psi}', \boldsymbol{\zeta}, \boldsymbol{\zeta}')$ into consideration, we immediately obtain a sufficient condition for the symmetric periodic motions (3.2) to be continuable: $\omega_1/\omega + l_s \neq l \in \mathbb{Z}$. Since a zero solution for ζ exists and the continuation is unique, we obtain plane orbits.

Theorem 1. If $\mu \leqslant \mu^*$, the many-body problem has solutions in which the bodies $\mathbf{M}_s$ $(s = 1, \ldots, n)$ describe closed plane orbits in a system of coordinates $\xi^*\eta^*\zeta$ rotating at a constant angular velocity ω_1. These orbits are symmetric about the ξ^* axis, the period of the motion of $\mathbf{M}_j$ $(j = 2, \ldots, n)$ in orbit around $\mathbf{M}_0$ equals $T_j = 2\pi/|l_j\omega|$ (where $l_j \in \mathbb{Z}, l_j \neq 0$, with l_j distinct), while the body $\mathbf{M}_1$ oscillates with period $T_1 = 2\pi/\omega$ about a relative position of equilibrium on the ξ^* axis at a distance a_1 from $\mathbf{M}_0$. As $\mu \to 0$ the orbits become concentric circles about $\mathbf{M}_0$, of radii $a_s = [f(m_0 + m_s)/\omega_s^2]^{1/3}$, where $|\omega_s| = |\omega_1 + l_s\omega|$ are the mean motions of the bodies M_s in orbits in the fixed space, with $\omega_s/\omega + l_s \neq l \in \mathbb{Z}$.

In the fixed space the steady motions are described by functions

$$\xi_s = a_s[1 + p_s(\tau)] \cos[(\omega_1 / \omega + l_s)\tau + \psi_s(\tau)]$$

$$\eta_s = a_s[1 + p_s(\tau)] \sin[(\omega_1 / \omega + l_s)\tau + \psi_s(\tau)], \quad \zeta_s = 0 \quad (s = 1, \ldots, n)$$

where $p_s(\tau)$, $\psi_s(\tau)$ are 2π-periodic functions of τ. Nevertheless, this solution is not periodic, as the numbers ω_1 and ω are rationally independent.

In a system of coordinates rotating at angular velocity ω_1, the bodies $\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_n$ will be situated at times which are multiples of π/ω along a single fixed straight line—the ξ_* axis. Obviously, at these times the bodies form a straight line in the fixed system of coordinates too (see Fig. 1). In that case, however, the straight line itself is displaced by an angular distance that is a multiple of $\Delta = \omega_1\pi/\omega$.

Thus, the steady periodic motions point to a possible way of explaining the resonances property of the Solar system and the phenomenon of "parades of planets".

The first approximation of the problem with respect to μ is

$$c_s' + \mu \sum_{j=1}^{n} \frac{k_j^* a_j}{\omega^2 r_{sj}^3 a_s} \sin[(l_s - l_j)\tau] = 0, \quad \psi_s' = c_s - 2\omega_s^* p_s$$

$$p_s'' + \omega_s^{*2} p_s - 2c_s + \mu \sum_{\substack{j=1 \\ j \neq s}}^{n} \frac{k_j^* \{a_s - a_j \cos[(l_s - l_j)\tau]\}}{\omega^2 r_{sj}^3 a_s} = 0 \quad (s = 1, \ldots, n)$$

Integrating the equations for c_s, we obtain

$$c_s = \frac{\mu}{\omega^2 a_s} \sum_{\substack{j=1 \\ j \neq s}}^{n} \frac{k_j^*}{(l_s - l_j) r_{sj}} + \text{const}$$

Then the remaining equations yield p_s and ψ_s as elliptic functions of τ.

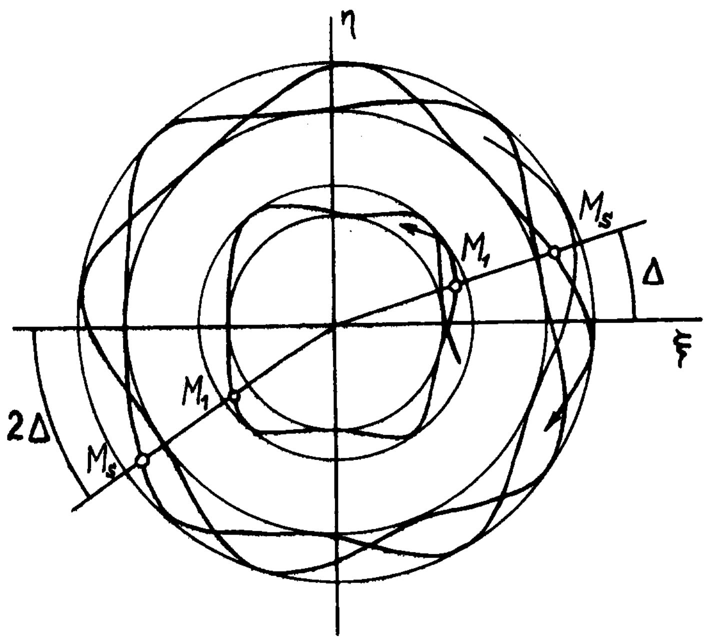

Fig. 1.

In the generating system, the variable c_s represents the variation of the area constant in the problem of two bodies $\mathbf{M}_0$ and $\mathbf{M}_s$ and maintains a constant value. When $\mu \neq 0$ and the solution is symmetric and periodic, c_s is a periodic function of τ, and the oscillations of c_s take place about the zero average. In that case

$$\sum_{s=1}^{n} a_s c_s k_s^* = 0$$

reflecting the conservation of angular momentum.

4. PERIODIC SOLUTIONS OF THE SECOND KIND

When $\mu = 0$ system (2.1) splits into n subsystems (indexed by s), each of which describes the unperturbed Keplerian motion of two bodies $\mathbf{M}_0$ and $\mathbf{M}_s$. The orbits are second-order curves

$$\rho_s(\theta_s) = \frac{\lambda_s}{1 + e_s \cos\theta_s}, \quad \rho_s^2 \theta_s^{\cdot} = c_s(\text{const}), \quad \lambda_s = \frac{c_s^2}{k_s}, \quad e_s = \frac{\sqrt{k_s^2 + h_s^2 c_s^2}}{k_s} \tag{4.1}$$

where c_s and h_s are the area and energy constants in the sth problem. The motions occur along ellipses if $0 < e_s < 1$ for all s. In symmetric periodic motions, the axes of the ellipses may lie along either the ξ_s or the η_s axes, simultaneously for all s (Fig. 2). Thus, the relation between the Cartesian and polar coordinates is given by one of the formulae

$$\xi_s = \rho_s(\theta_s)\begin{cases}\cos\theta_s \\ \sin\theta_s\end{cases}, \quad \eta_s = \rho_s(\theta_s)\begin{cases}\sin\theta_s \\ \cos\theta_s\end{cases} \tag{4.2}$$

The elliptic motion (4.2) is periodic in the sth subsystem. For the entire generating system, solution (4.1) forms a family of symmetric conditionally-periodic motions which is $2n$-parametric with respect to the initial data (c_s, h_s). Some of these motions are periodic—those for which

$$n_s = \sqrt{k_s} \,/\, a_s^{3/2} = l_s\omega \quad (l_s \in \mathbb{Z}), \quad \lambda_s = a_s(1 - e_s^2) \quad (s = 1, \ldots, n) \tag{4.3}$$

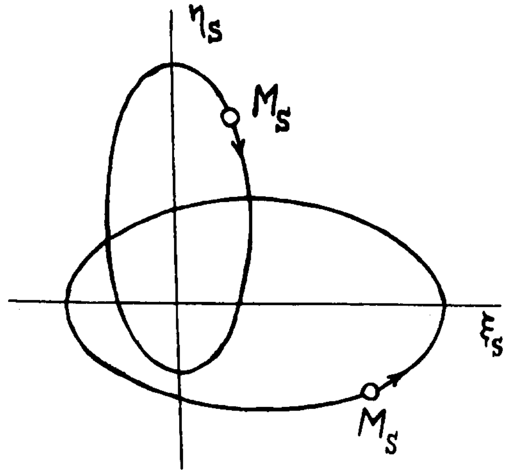

Fig. 2.

where a_s are the semi-major axes of the ellipses and ω is a positive quantity. These relations mean that the mean motions along the generating ellipses relate to one another as natural numbers.

The $n-1$ conditions (4.3) are imposed on the $2n$ constants c_s, h_s. Hence we have an $(n + 1)$-parametric family of periodic motions, depending on the initial data. We also note that conditions (4.3) involve only the semi-axes a_s of the ellipses, without affecting the eccentricities e_s, while the direction of the motion is determined individually along each ellipse.

Periodic solutions obtained by continuing the elliptic solutions (4.1) to non-zero values of the parameter μ are known as periodic solutions of the second kind.

Let us consider the upper one of the two possibilities in (4.2). Then, in the neighbourhood of the generating solution (4.1)

$$\xi_s + i\eta_s = \rho_s(\theta_s)\, e^{i\theta_s}(1+p_s), \quad \xi_s - i\eta_s = \rho_s e^{-i\theta_s}(1+q_s) \quad (s=1,\dots,n)$$

(in the other case one should simply interchange ξ_s and η_s). This yields equations for p_s and q_s, through which one can estimate the domain of analyticity and the Lipschitz constant. The full equations are rather cumbersome; the part linear in p_s and q_s, and independent of μ is

$$\begin{gathered}\frac{d^2 p_s}{d\theta_s^2} + 2i\frac{dp_s}{d\theta_s} - \frac{3\rho_s}{2\lambda_s}(p_s+q_s) + \ldots = 0 \\ \frac{d^2 q_s}{d\theta_s^2} - 2i\frac{dq_s}{d\theta_s} - \frac{3\rho_s}{2\lambda_s}(p_s+q_s) + \ldots = 0 \quad (s=1,\dots,n)\end{gathered} \tag{4.4}$$

The unwritten terms are 2π-periodic functions of $\theta_1, \dots, \theta_n$, and if conditions (4.3) are satisfied they are 2π-periodic in the single variable $\theta = \omega t$, which is indeed taken as a new independent variable. When that is done $\theta_s = l_s\theta + f_s(l_s\theta)$, where f_s is a Fourier series in θ_s. With that in mind, it is nevertheless convenient or our purposes to retain the angle θ_s as the variable for the sth subsystem.

Elementary reduction of system (4.4) yields the final system

$$\begin{gathered}\frac{dc_s}{d\theta_s} + \ldots = 0, \quad \frac{d(p_s - q_s)}{d\theta_s} + 2i(p_s+q_s) - c_s + \ldots = 0 \\ \frac{d^2(p_s+q_s)}{d\theta_s^2} + \left[4 - \frac{3}{1+e_s\cos\theta_s}\right](p_s+q_s) + 2ic_s + \ldots = 0\end{gathered} \tag{4.5}$$

which is invariant under the change of variables $(\theta_s, c_s, p_s - q_s, p_s + q_s) \to (-\theta_s, c_s, -(p_s - q_s), p_s + q_s)$,

($s = 1, \dots, n$). System (4.5) in the linear approximation splits into n subsystems. Moreover, for each s the construction of the general solution of the above part requires an analysis of a single second-order equation

$$\frac{d^2x}{d\theta^2} + g(\theta)\frac{dx}{d\theta} + f(\theta) = 0, \quad f(\theta+2\pi) = f(\theta), \quad g(\theta+2\pi) = g(\theta) \tag{4.6}$$

which is invariant under the change of variable $(\theta, x, x') \to (-\theta, x, -x')$, $x' = dx/d\theta$.

For each solution $x = x(\theta), x' = x'(\theta)$ of Eq. (4.6) there is also a solution $x = x(-\theta), x' = x'(-\theta)$, and by virtue of the linearity of the functions

$$(1) \quad x = x(\theta) + x(-\theta), \quad x' = x'(\theta) - x'(-\theta), \quad (2) \quad x = x(\theta) - x(-\theta), \quad x' = x'(\theta) + x'(-\theta)$$

are also solutions. For the first of these solutions we have $x'(0) = 0$, and for the second, $x(0) = 0$. Therefore, if the characteristic exponents of (4.6) are $\pm\kappa$, a fundamental system of solutions of Eq. (4.6), given that the initial data from the identity matrix, is

$$\begin{Vmatrix} A_+(\theta,\kappa) & A_-(\theta,\kappa) \\ A_-^*(\theta,\kappa) & A_+^*(\theta,\kappa) \end{Vmatrix} \tag{4.7}$$

$$A_\pm(\theta,\kappa) = \alpha(\theta)\,e^{\kappa\theta} \pm \alpha(-\theta)\,e^{-\kappa\theta}, \qquad A_\pm^*(\theta,\kappa) = \alpha(\theta)\,e^{\kappa\theta} \pm \alpha^*(-\theta)\,e^{-\kappa\theta}$$

where the 2π-periodic functions $\alpha(\theta)$, $\alpha^*(\theta)$ equal $1/2$ at zero.

Now let $\pm\kappa_s$ be the characteristic exponents of the equations for $p_s + q_s$ in system (4.5). Since the vectors $\mathbf{p}$ and $\mathbf{q}$ of Eq. (1.3) in this case have the form $\mathbf{p} = (c_s, p_s + q_s)^T$ and $\mathbf{q} = (p_s - q_s, p_s' + q_s')^T$, it follows that if $\kappa_s \neq \mathrm{N}i$ ($N \in \mathbb{Z}$), then

$$\mathbf{p}^+ = \begin{Vmatrix} 1 & 0 \\ \alpha_s^{**}(\theta_s) & A_+(\theta_s,\kappa_s) \end{Vmatrix}$$

$$\mathbf{q}^- = \begin{Vmatrix} \beta_s^*(\theta_s) + \theta_s & \Gamma_-^*(\theta_s,\kappa_s) \\ \gamma_s^{**}(\theta_s) & A_-^*(\theta_s,\kappa_s) \end{Vmatrix}$$

$$A_+(\theta_s,\kappa_s) = \alpha_s(\theta_s)\exp(\kappa_s\theta_s) + \alpha_s(-\theta_s)\exp(-\kappa_s\theta_s)$$

$$A_-^*(\theta_s,\kappa_s) = \alpha_s^*(\theta_s)\exp(\kappa_s\theta_s) - \alpha_s^*(-\theta_s)\exp(-\kappa_s\theta_s)$$

$$\Gamma_-^*(\theta_s,\kappa_s) = \gamma_s^*(\theta_s)\exp(\kappa_s\theta_s) - \gamma_s^*(-\theta_s)\exp(-\kappa_s\theta_s)$$

where all functions are 2π-periodic in θ_s and, in addition, $\beta_s^*(\theta_s)$ are odd functions of θ_s. Therefore, $\beta_s^*(l_s\pi) = 0$ and

$$\det q^-(\pi) = \pi^n \prod_{s=1}^{n} \alpha_s^*(0)\{\exp(\kappa_s l_s\pi) - \exp(-\kappa_s l_s\pi)\}$$

Consequently, if $\kappa_s \neq \pm i\,\nu_s/l_s$ ($\nu_s = 1, \dots, l_s$: $s = 1, \dots, n$), the conditions for symmetric generating solutions (4.1) to be continuable to non-zero μ are satisfied.

It is obvious from the matrix (4.7) that in order to determine the characteristic exponents one has to compute the value at $\theta = 2\pi$ of only one particular solution with initial data, say, $x(0) = 1, x'(0) = 0$. Then $2A = 2x(2\pi) = e^{2\pi\kappa} + e^{-2\pi\kappa}$. Thus the solutions can be continued if $A_s \neq \cos(2\pi\nu_s/l_s)$ for all s. Figure 3 shows a plot of A_s versus the eccentricity e_s. Obviously, the quantities A_s do not take critical values except at a finite number of points.

Theorem 2. If $\mu \leq \mu^*$, the $(n + 1)$-body problem has a family of plane symmetric periodic orbits, which depend on the $n + 1$ initial data as parameters, which arise in a unique manner from the elliptic generating solutions (4.1). An exceptional situation occurs at a finite number of eccentricities, when such symmetric periodic orbits may fail to exist when $\mu \neq 0$.

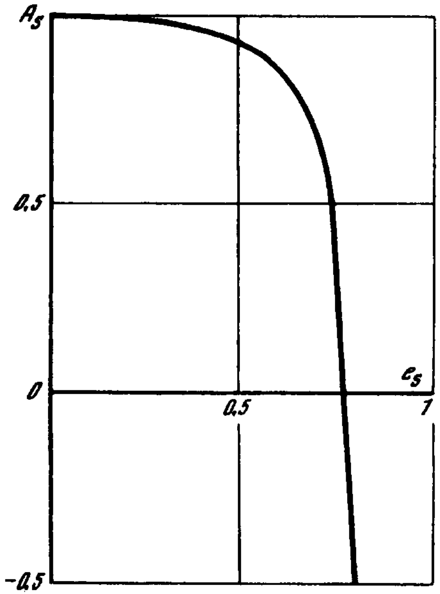

Fig. 3.

All the bodies, moving in symmetric periodic orbits, intersect the set of fixed points of the automorphism. If this set is the ξ axis, then the bodies $\mathbf{M}_s$ line up periodically, at intervals $T = \pi/\omega$, along the ξ axis, forming a "parade of planets". When this happens the motion of the system is, of course, resonant: the mean motions relate to one another as integers.

Cases in which the eccentricities e_s of the generating solution (4.1) are small are of particular interest. The characteristic exponents are then [6]

$$\kappa_s = i(4 - 3/\sqrt{1-e_s^2}) + \ldots$$

and the conditions for the motions (4.1) to be continuable for non-zero μ are clearly satisfied.

Theorem 3. For sufficiently small values of the eccentricities e_s of the generating elliptic solution (4.1), symmetric periodic orbits exist for $\mu \neq 0$ also.

5. THE SUN–PLANET–SATELLITES SYSTEM

The results just established are not only applicable to Sun–planet type systems. Indeed, let us rewrite (2.1) as

$$\ddot{\xi}_1 + \frac{k_1\xi_1}{r_{01}^3} + \mu \sum_{j=2}^{n} k_j^* \left[\frac{\xi_1-\xi_j}{r_{1j}^3} + \frac{\xi_j}{r_{0j}^3}\right] = 0$$

$$\ddot{\xi}_s + \frac{k_s\xi_s}{r_{0s}^3} + \frac{k\mu_1}{r_{s1}^3}(\xi_s-\xi_1) + \mu\sum_{j=2}^{n} k_j^*\left[\frac{\xi_s-\xi_j}{r_{sj}^3}+\frac{\xi_j}{r_{0j}^3}\right]=0 \quad (s=2,\ldots,n)$$

where now $\mu = \max_{2 \leqslant s \leqslant n} \{\mu_s\}$ and μ_1 may be substantially greater than $1 + \mu_s$ ($s = 2, \ldots, n$). Then when $\mu = 0$ the equations for $\mathbf{M}_1$ admit of a solution in which $\mathbf{M}_1$ describes a circle a_1 at constant angular velocity ω_1 ($\omega_1^2 a_1^3 = k\,\mu_1$). From the mechanical point of view, it is clear that for sufficiently large a_1 (small ω_1) the effect of $\mathbf{M}_1$ on the motion of the system of bodies consisting of a "planet" $\mathbf{M}_0$ and "satellites" $\mathbf{M}_2, \ldots, \mathbf{M}_n$ will be small; this should be expressed in mathematical terms by the smallness of the parameter.

To prove this, let us change to variables p_s, ψ_s, ζ_s, assuming that the constants a_s, ω_s ($s = 1, \ldots, n$) are given by (3.2). This gives a system (3.3) with summation running from $j = 2$ to n, while the term with $j = 1$ determines the effect of $\mathbf{M}_1$ on $\mathbf{M}_s$ ($s = 2, \ldots n$). Assuming now that $a_s \ll a_1$ ($s = 2, \ldots, n$), we set $a_s^* = a_s/a_1$. Then the factor multiplying the term with $j = 1$ is $[\mu_1/(1 + \mu_1)]\,(\omega_1/\omega_s)^2$. Therefore, if $(\omega_1/\omega_s)^2$ is of the same order of magnitude as μ, the term with $j = 1$ may also be included in the perturbations.

When $\mu = 0$ this system admits of the trivial solution $\mathbf{p} = \mathbf{0}$, $\boldsymbol{\psi} = \mathbf{0}$, $\boldsymbol{\zeta} = \mathbf{0}$. These solutions can be continued to non-zero μ as in Section 3.

Thus, a Sun–planet–satellites system admits of resonant periodic solutions such that, in a system of coordinates rotating at angular velocity ω_1 around the planet, the satellites describe symmetric closed orbits which are nearly concentric circles. When $n = 2$ this yields symmetric orbits of the moon in the Sun–Earth–Moon system, constructed previously [7] within the framework of the limited three-body problem.

In conclusion, we note that elliptic generating orbits in a Sun–planet–satellites system can also be continued.

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